

THE CAUCHY PROBLEM FOR THE HOMOGENEOUS MONGE-AMPÈRE EQUATION, I. TOEPLITZ QUANTIZATION

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ABSTRACT. The Cauchy problem for the homogeneous (real and complex) Monge-Ampère equation (HRMA/HCMA) arises from the initial value problem for geodesics in the space of Kähler metrics. It is an ill-posed problem. We conjecture that, in its lifespan, the solution can be obtained by Toeplitz quantizing the Hamiltonian flow defined by the Cauchy data, analytically continuing the quantization, and then taking a kind of logarithmic classical limit. In this article, we prove that in the case of torus invariant metrics (where the HCMA reduces to the HRMA) this “quantum analytic continuation potential” coincides with the well-known Legendre transform potential, and hence solves the equation as long as it is smooth. In the sequel [RZ2] we prove that the Legendre transform potential ceases to solve the HRMA after that time.

1. INTRODUCTION

This article is the first in a series whose aim is to study existence, uniqueness and regularity of solutions of the initial value problem (IVP) for geodesics in the space of Kähler metrics in a fixed class. It is a special case of the Cauchy problem for the HCMA (homogeneous complex Monge-Ampère equation). Unlike the much-studied Dirichlet problem little has been proven for the Cauchy problem for the Monge-Ampère equation, and there is currently no known method to solve it for smooth Cauchy data. Indeed, it is an ill-posed problem and one does not expect global in time solutions to exist for ‘most’ initial data. The goal is thus to determine which initial data give rise to global solutions, especially those of relevance in geometry (‘geodesic rays’) and to determine the lifespan T_{span} of solutions for general initial data. In this article, we propose a general solution to the IVP for the geodesic equation on a polarized projective Kähler manifold, valid for the lifespan of the solution, in terms of a Toeplitz quantization and its analytic continuation. This conjectural solution, which we call the “quantum analytic continuation potential,” is defined as the logarithmic limit of a canonical sequence of subsolutions of the HCMA obtained from the analytic continuation in time of the Toeplitz quantization of the Cauchy data.

Our first goal in this series is to show that the conjectured solution is indeed a solution to the IVP for geodesics, as long as one exists, when the Kähler manifold (M, ω) has an $(S^1)^n$ symmetry with $n = \dim M$. In such cases (including toric

Kähler manifolds and Abelian varieties), the HCMA reduces to the HRMA (homogeneous real Monge-Ampère equation). Even in this setting, the problem is rather involved, and its different aspects are treated separately in the different articles of the series. In this article, we prove that in the $(S^1)^n$ -invariant case, the quantum analytic continuation potential is a Lipschitz continuous subsolution that is a smooth solution of the HRMA until the ‘convex lifespan’ $T_{\text{span}}^{\text{cvx}}$ of the problem (see Definition 3.1). In the sequel [RZ2], we show that the quantum analytic continuation potential fails to solve the equation even in a weak sense after the convex lifespan. In [RZ3], we characterize the smooth lifespan of the HCMA. In particular, for the HRMA, we show that the smooth lifespan T_{span}^∞ (see Definition 2.2) of the Cauchy problem equals the convex lifespan. Hence the directions of smooth geodesic rays are those with infinite convex lifespan.

This article and the next one [RZ2] are devoted mainly to the HRMA and to Kähler manifolds with symmetry. However, the quantum analytic continuation potential constructed in this article (see §2 and §5), and the characterization of the smooth lifespan in [RZ3], apply to the HCMA and to general Kähler manifolds. In addition, we believe that the rest of the methods developed here have natural extensions at least to the case of Riemann surfaces.

Our study is to a large extent motivated by applications to Kähler geometry, that we now briefly describe. Let (M, J, ω) denote a closed compact Kähler manifold of complex dimension n . Consider the infinite-dimensional space

$$\mathcal{H}_\omega = \{\varphi \in C^\infty(M) : \omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}, \quad (1)$$

of Kähler metrics in a fixed Kähler class equipped with the Riemannian metric [M, S, D1]

$$g_{L^2}(\zeta, \eta)_\varphi := \frac{1}{V} \int_M \zeta \eta \omega_\varphi^n, \quad \varphi \in \mathcal{H}_\omega, \quad \zeta, \eta \in T_\varphi \mathcal{H}_\omega \cong C^\infty(M). \quad (2)$$

One may show that covariant differentiation on $(\mathcal{H}_\omega, g_{L^2})$ is given by

$$D_c e = \dot{e} - \frac{1}{2} g_\varphi(\nabla c, \nabla e), \quad (3)$$

where $\gamma(s)$ is a curve in \mathcal{H}_ω with $\gamma(0) = \varphi$, $\dot{\gamma}(0) = c \in T_\varphi \mathcal{H}_\omega$ and $e(s) = e(\gamma(s))$ is a vector field on \mathcal{H}_ω along γ . Here g_φ is the Riemannian metric associated to ω_φ and ∇ is the Levi-Civita connection of g_φ . Hence, geodesics of $(\mathcal{H}_\omega, g_{L^2})$ are maps φ from a connected subset I of \mathbb{R} to \mathcal{H}_ω , equivalently functions on $I \times M$, that satisfy the equation

$$\ddot{\varphi} - \frac{1}{2} g_\varphi(\nabla \dot{\varphi}, \nabla \dot{\varphi}) = 0, \quad \text{on } (I \setminus \partial I) \times M. \quad (4)$$

Extend φ in a trivial manner to $(I \setminus \partial I) \times \mathbb{R} \times M$, i.e., by setting φ to be \mathbb{R} -invariant, and denote by π_2 the projection map from this product to M , and by $\tau = s + \sqrt{-1}t$ the holomorphic coordinate on $(I \setminus \partial I) \times \mathbb{R}$. It was observed by Semmes and Donaldson that

$$\frac{1}{n+1} (\pi_2^* \omega + \sqrt{-1} \partial \bar{\partial} \varphi)^{n+1} = \quad (5)$$

$$(\ddot{\varphi} - \frac{1}{2} g_\varphi(\nabla \dot{\varphi}, \nabla \dot{\varphi})) \sqrt{-1} d\tau \wedge d\bar{\tau} \wedge \omega_\varphi^n, \quad \text{on } (I \setminus \partial I) \times \mathbb{R} \times M.$$

Therefore, when φ is regular enough, the geodesic equation is equivalent to the homogeneous complex Monge-Ampère (HCMA) equation on the product of a Riemann surface with M .

The initial value problem is the problem of defining the exponential map of \mathcal{H}_ω . Although the Cauchy problem is ill-posed for the HCMA, infinite geodesic rays are expected to play an important role in Kähler geometry and this is one motivation to study the IVP (see [AT, Ch, CTa, CT, D1, M, PS2, PS3, S, Su] for relevant Kähler geometry background). Yet the ill-posedness makes the Cauchy problem very different from the Dirichlet problem corresponding to geodesics connecting two given end-points, whose existence and regularity was first studied extensively by Chen [Ch], Donaldson [D2], and Chen-Tian [CT]. As observed by Mabuchi, Semmes, and Donaldson, \mathcal{H}_ω is formally an infinite dimensional symmetric space of the type $G^\mathbb{C}/G$ where G is the group of Hamiltonian diffeomorphisms of (M, ω) . Hence its geodesics should be given by certain one-parameter subgroups of $G^\mathbb{C}$, which correspond to analytic continuations in time of Hamiltonian orbits. To a large extent, the Kähler quantization method of this article is an attempt to put these formal arguments on a rigorous basis.

The article is organized as follows. In Section 2 we describe our approach to the IVP using an analytic continuation of Toeplitz quantization. Our main results are stated in Section 3, and in Section 4 we recall some background. In Section 5 we construct the quantization of the Hamiltonian flow. The results in this Section hold on an arbitrary projective Kähler manifold. In Section 6 we specialize to the setting of a toric or Abelian variety where we construct a second quantization of the Hamiltonian flow and compare the two quantizations and their analytic continuations. In Section 7 we complete the proof of our main result (Theorem 1), showing that the analytic continuations of the quantizations converge to the Legendre transform potential and solve the Cauchy problem until the convex lifespan.

2. A QUANTUM MECHANICAL APPROACH TO MONGE-AMPÈRE

In this section we define the *quantum analytic continuation potential* and state the general conjecture that it solves the IVP for geodesics in $(\mathcal{H}_\omega, g_{L^2})$, to the extent possible, in the case of projective Kähler manifolds. The definition is inspired by two prior constructions and is largely aimed at reconciling them.

The first is a heuristic analytic continuation argument due to Semmes and Donaldson [S, ?]: Let $\dot{\varphi}_0$ be a smooth function on M , considered as a tangent vector in $T_{\varphi_0}\mathcal{H}_\omega$. Let $X_{\dot{\varphi}_0}^{\omega_{\varphi_0}} \equiv X_{\dot{\varphi}_0}$ denote the Hamiltonian vector field associated to $\dot{\varphi}_0$ and (M, ω_{φ_0}) and let $\exp tX_{\dot{\varphi}_0}$ denote the associated Hamiltonian flow. Then let $\exp \sqrt{-1}sX_{\dot{\varphi}_0}$ “be” its analytic continuation in time to the Hamiltonian flow at “imaginary” time $\sqrt{-1}s$. Then “define” the *classical analytic continuation potential* φ_s with initial data $(\varphi_0, \dot{\varphi}_0)$ by

$$(\exp \sqrt{-1}sX_{\dot{\varphi}_0})^*\omega_0 - \omega_0 = \sqrt{-1}\partial\bar{\partial}\varphi_s. \quad (6)$$

Then φ_s “is” the solution of the initial value problem. We use quotes since there is no obvious reason why $\exp tX_{\dot{\varphi}_0}$, a rather arbitrary smooth Hamiltonian flow, should admit an analytic continuation in t for any length of time. When the analytic continuation does exist, e.g., if ω_{φ_0} and $\dot{\varphi}_0$ are real analytic, then φ_s solves the initial value problem for the Monge-Ampère equation for s in some (usually) small time interval.

The second construction uses finite dimensional approximations deriving from Kähler quantization. The idea is to approximate the space \mathcal{H}_ω by finite-dimensional spaces of Bergman (or Fubini-Study) metrics induced by holomorphic embeddings of M into \mathbb{P}^N using bases of holomorphic sections $s \in H^0(M, L^k)$ of high powers of a polarizing line bundle. Following an original idea of Yau and Tian, such embeddings were used in [T, C, Z3] to approximate individual metrics. Phong-Sturm [PS1, PS2] then introduced a Kähler quantization method to approximate geodesic segments with fixed end-points by geodesics in the space of Bergman metrics. They also used the method to define geodesic rays from test configurations. Further work on Bergman approximations to geodesics, as well as more general harmonic maps, are due to Berndtsson, Chen-Sun, Feng, Song-Zelditch, and others [B1, B2, CS, Fe, RZ1, SoZ1, SoZ2].

Our approach combines the two as follows: we define the analytic continuation of $\exp tX_{\dot{\varphi}_0}$ by quantizing this Hamiltonian flow, by analytically continuing the quantum flow, and then by taking a kind of logarithmic classical limit of its Schwartz kernel.

Consider the Hilbert spaces of sections $L^2(M, L^N)$, $N \in \mathbb{N}$, associated to powers of a Hermitian line bundle (L, h_0) polarizing (M, ω_{φ_0}) , and the corresponding orthogonal projection operators

$$\Pi_N \equiv \Pi_{N, \varphi_0} : L^2(M, L^N) \rightarrow H^0(M, L^N),$$

onto the Hilbert subspaces $H^0(M, L^N)$ of holomorphic sections. These Hilbert subspaces allow one to ‘quantize’ (M, ω_{φ_0}) . In order to quantize the Hamiltonian flow of $X_{\dot{\varphi}_0}$ on (M, ω_{φ_0}) we use the method of Toeplitz quantization. Namely, we consider the operators

$$\Pi_N \circ \dot{\varphi}_0 \circ \Pi_N,$$

where here $\dot{\varphi}_0$ denotes the operator of multiplication by $\dot{\varphi}_0$. We will usually omit the composition symbols and denote these by $\Pi_N \dot{\varphi}_0 \Pi_N$. These are zero-order self-adjoint operators. Define the associated one-parameter subgroups of unitary operators

$$U_N(t) := \Pi_N e^{\sqrt{-1}tN\Pi_N \dot{\varphi}_0 \Pi_N} \Pi_N \quad (7)$$

on $H^0(M, L^N)$.

A key observation is that there is no obstruction to analytically continuing the quantization: each $U_N(t)$ admits an analytic continuation in time t and induces the imaginary time semi-group

$$U_N(\sqrt{-1}s) : H^0(M, L^N) \rightarrow H^0(M, L^N), \quad U_N(\sqrt{-1}s) \in GL(H^0(M, L^N), \mathbb{C}). \quad (8)$$

The main idea of this article is that the analytic continuation of $\exp tX_{\dot{\varphi}_0}$ can be constructed by taking a non-standard kind of logarithmic classical limit of the analytic continuation of its quantization. We do this by considering the Schwartz kernel $U_N(-\sqrt{-1}s)(z, w)$ of this operator with respect to the volume form $(N\omega_{\varphi_0})^n$.

DEFINITION 2.1. *Set*

$$\varphi_N(s, z) := \frac{1}{N} \log U_N(-\sqrt{-1}s, z, z). \quad (9)$$

We define the quantum analytic continuation potential φ_∞ by

$$\varphi_\infty(s, z) := \lim_{l \rightarrow \infty} (\sup_{N \geq l} \varphi_N)_{\text{reg}}(s, z).$$

Here, $u_{\text{reg}}(z_0) := \lim_{\epsilon \rightarrow 0} \sup_{|z - z_0| < \epsilon} u(z)$ denotes the upper semi-continuous regularization of u . The limit on the right hand side exists and is $\pi_2^* \omega$ -plurisubharmonic, since it is a limit of a sequence of decreasing $\pi_2^* \omega$ -psh functions ([De2], §I.5).

This limit is quite different from the semi-classical limits studied in Toeplitz quantization, because the analytic continuation in time destroys the Toeplitz structure of the kernel. Moreover, the logarithmic asymptotics of the Schwartz kernel is quite unrelated to symbol asymptotics. One may think of it as extracting an analytic continuation of the ‘phase function’ of the Toeplitz operator; the ‘symbol’ of the Toeplitz operator is irrelevant.

Denote by

$$S_T := [0, T] \times \mathbb{R}$$

the (vertical) strip of width T in \mathbb{C} . The IVP for geodesics is equivalent to the following Cauchy problem for the homogeneous complex Monge-Ampère equation:

$$\left\{ \begin{array}{ll} (\pi_2^* \omega + \sqrt{-1} \partial \bar{\partial} \varphi)^{n+1} = 0 & \text{on } S_T \times M, \\ \varphi(0, s, \cdot) = \varphi_0(\cdot) & \text{on } \{0\} \times \mathbb{R} \times M, \\ \frac{\partial \varphi}{\partial s}(0, s, \cdot) = \dot{\varphi}_0(\cdot) & \text{on } \{0\} \times \mathbb{R} \times M, \end{array} \right. \quad (10)$$

Note here that the complex Monge-Ampère operator is well-defined on bounded plurisubharmonic functions [BT1, BT2].

DEFINITION 2.2. *We define the smooth lifespan (respectively, lifespan) of the Cauchy problem (10) to be the supremum over all $T \geq 0$ such that (10) admits a smooth (respectively $\pi_2^* \omega$ -psh) solution. We denote the smooth lifespan (respectively, lifespan) for the Cauchy data $(\omega_{\varphi_0}, \dot{\varphi}_0)$ by $T_{\text{span}}^\infty \equiv T_{\text{span}}^\infty(\omega_{\varphi_0}, \dot{\varphi}_0)$ (respectively, $T_{\text{span}} \equiv T_{\text{span}}(\omega_{\varphi_0}, \dot{\varphi}_0)$).*

DEFINITION 2.3. *Define the quantum lifespan T_{span}^Q of the Cauchy problem (10) to be supremum over all $T \geq 0$ such that the quantum analytic continuation potential φ_∞ solves the HCMA (10).*

We pose the following conjecture, which would give a general method to solve the ill-posed Cauchy problem for the HCMA to the extent possible.

CONJECTURE 2.4. *The quantum analytic continuation potential φ_∞ solves the HCMA (10) for as long as it admits a solution. In other words, $T_{\text{span}}^Q = T_{\text{span}}$.*

As mentioned above, the key difficulty in the analysis is that although $U_N(t, z, w)$ is a standard Toeplitz Fourier integral operator quantizing the Hamilton flow of $\dot{\varphi}_0$, its analytic continuation $U_N(-\sqrt{-1}s, z, z)$ lies outside the class of complex Fourier integral operators, and it is difficult to analyze its logarithmic asymptotics or to determine how regular the limit should be. The toric setting provides a testing ground where it is possible to make a complete analysis. We only give the details for toric Kähler manifolds, but as in [Fe], the same methods apply to Abelian varieties.

3. STATEMENT OF RESULTS

The main results of this article concern the Cauchy problem for the HRMA. While the Dirichlet problem for the HRMA has been extensively studied (see [RT, CNS, GTW, Gz] and references therein), the Cauchy problem has not been systematically investigated. We are only aware of [BB] that proves uniqueness of C^3 solutions for the Cauchy problem for the more general HCMA, of [Fo1, Fo2], where a sufficient condition on the Cauchy data is given for existence of a smooth short-time solution of HRMA depending on the Cauchy hypersurface (for our Cauchy hypersurface, the existence of a smooth short-time solution is not an issue, since it follows independently from a classical Legendre duality argument), and of [U] where an explicit formula is derived for smooth solutions of the 2-dimensional HRMA.

In general, the HRMA can be viewed as a special case of the HCMA under the presence of sufficient symmetry. In the setting of the HCMA (5) corresponding to the IVP for geodesics, the reduction to a HRMA precisely corresponds to restricting from a general projective variety to a toric or Abelian one. Let us now describe briefly this geometric setting, concentrating on the toric case (for more background see §4.2).

A toric Kähler manifold is a Kähler manifold (M, J, ω) that admits a holomorphic action of a complex torus $(\mathbb{C}^*)^n$ with an open dense orbit, and for which the Kähler form ω is toric, i.e., invariant under the action of the real torus

$$\mathbf{T} := (S^1)^n.$$

We assume that the Cauchy data $(\omega_{\varphi_0}, \dot{\varphi}_0)$ is toric, and consider the IVP for geodesics in the space of torus-invariant Kähler metrics. Over the open orbit

$$M_o \cong (\mathbb{C}^*)^n \cong \mathbb{R}^n \times \mathbf{T}$$

the Kähler form ω_{φ_0} is exact and \mathbf{T} -invariant and so we let ψ_0 be a smooth strictly convex function on \mathbb{R}^n satisfying

$$\omega_{\varphi_0}|_{M_o} = \sqrt{-1}\partial\bar{\partial}\psi_0. \quad (11)$$

Here $[\omega]$ is any integral Kähler class in $H^2(M, \mathbb{Z})$. The initial velocity $\dot{\varphi}_0$ is also \mathbf{T} -invariant, and so it induces, by restriction to the open orbit, a smooth bounded function on \mathbb{R}^n , that we denote by $\dot{\psi}_0$. Analytically, the IVP is then equivalent to studying the following HRMA for a convex function ψ on $[0, T] \times \mathbb{R}^n$,

$$\left\{ \begin{array}{ll} \text{MA } \psi = 0, & \text{on } [0, T] \times \mathbb{R}^n, \\ \psi(0, \cdot) = \psi_0(\cdot), & \text{on } \mathbb{R}^n, \\ \frac{\partial \psi}{\partial s}(0, \cdot) = \dot{\psi}_0(\cdot), & \text{on } \mathbb{R}^n. \end{array} \right. \quad (12)$$

Here, MA denotes the real Monge-Ampère operator that can be defined as a Borel measure on convex functions

$$\text{MA } f := d \frac{\partial f}{\partial x^1} \wedge \cdots \wedge d \frac{\partial f}{\partial x^{n+1}}, \quad \text{for } f \text{ convex on } \mathbb{R}^{n+1},$$

and equals $\det \nabla^2 f \, dx^1 \wedge \cdots \wedge dx^{n+1}$ on C^2 functions [RT].

Let

$$P := \overline{\text{Im} \nabla \psi_0} \subset \mathbb{R}^n.$$

Recall that on a symplectic toric manifold the Legendre transform $f \mapsto f^*$ is a bijection between the set of \mathbf{T} -invariant Kähler potentials on the open orbit $M_o \cong (\mathbb{C}^n)^*$ of the (complex) torus action

$$\mathcal{H}(\mathbf{T}) := \{\psi \in C^\infty(\mathbb{R}^n) : \sqrt{-1} \partial \bar{\partial} \psi = \omega_\varphi|_{M_o} \text{ with } \varphi \in \mathcal{H}_\omega \text{ and } \overline{\text{Im} \nabla \psi} = P\},$$

and the set of symplectic potentials on the moment polytope $P \subset \mathbb{R}^n$

$$\mathcal{LH}(\mathbf{T}) := \{u \in C^\infty(P \setminus \partial P) \cap C^0(P) : u = \psi^* \text{ with } \psi \in \mathcal{H}(\mathbf{T})\}. \quad (13)$$

When the latter space is equipped with the standard $L^2(P)$ metric, this map is in fact an isometry and transforms the IVP geodesic equation (4) to the linear equation

$$\ddot{u} = 0, \quad u_0 = \psi_0^*, \quad \dot{u}_0 = -\dot{\psi}_0 \circ (\nabla \psi_0)^{-1}, \quad (14)$$

whose solution is given by

$$u_s := u_0 + s \dot{u}_0.$$

DEFINITION 3.1. *Define the convex lifespan of the Cauchy problem (12) as*

$$T_{\text{span}}^{\text{cvx}}(\psi_0, \dot{\psi}_0) := \sup \{s : \psi_0^* - s \dot{\psi}_0 \circ (\nabla \psi_0)^{-1} \text{ is convex on } P\}.$$

We note that $T_{\text{span}}^{\text{cvx}}$ is independent of the choice of ψ_0 satisfying (11).

At least as long as $s < T_{\text{span}}^{\text{cvx}}$, i.e., u_s is strictly convex and hence belongs to $\mathcal{LH}(\mathbf{T})$, it is well-known that the IVP for geodesics has an explicit solution,

$$\psi(s, x) = \psi_s(x) := (u_0 + s \dot{u}_0)^*(x), \quad s \in [0, T_{\text{span}}), \quad x \in \mathbb{R}^n. \quad (15)$$

For a review of this fact and references we refer to [RZ2]. We call ψ the *Legendre transform potential*.

What is less transparent is what happens when $s > T_{\text{span}}^{\text{cvx}}$. Firstly, it should be pointed out that, as defined in (15), ψ_s is finite for each $x \in \mathbb{R}^n$. Hence, it is

necessarily Lipschitz. Moreover, as we show in [RZ2], ψ_s is strictly convex, but not differentiable everywhere.

Denote by $\mathcal{H}^{0,1}(\mathbf{T})$ the closure of $\mathcal{H}(\mathbf{T})$ with respect to the $C^{0,1}$ -norm (this space contains also convex functions that are not strictly convex). The corresponding space of ω -psh (plurisubharmonic) functions will be denoted by $\mathcal{H}_\omega^{0,1}$. According to the previous paragraph, one has $\psi_s \in \mathcal{H}^{0,1}(\mathbf{T})$ for all $s > 0$. It therefore makes sense to consider ψ as an infinite ray in the interior of $\mathcal{H}^{0,1}(\mathbf{T})$.

Our main result in this article states that the sequence of level N quantum analytic continuation potentials φ_N defined by (9) converges uniformly to the Legendre transform potential ψ , and therefore the quantum analytic continuation potential φ_∞ of Definition 2.1 solves the HCMA for $T < T_{\text{span}}^{\text{cvx}}$.

THEOREM 1. *Let $\varphi := \psi - \psi_0$ be the one-parameter family of Lipschitz continuous ω -psh potentials associated to the Legendre transform potential ψ given by (15), and let φ_N be the quantum analytic continuation potentials given by (9). Then*

$$\lim_{N \rightarrow \infty} \varphi_N = \varphi$$

in $C^2([0, T] \times M)$ for $T < T_{\text{span}}^{\text{cvx}}$, and in $C^0([0, T] \times M)$ for $T \geq T_{\text{span}}^{\text{cvx}}$. In particular, the quantum analytic continuation potential coincides with the Legendre transform potential

$$\varphi_\infty = \varphi \in \mathcal{H}_\omega^{0,1}.$$

In the sequel, we prove that the quantum analytic continuation potential φ ceases to solve the HCMA (10) for any $T > T_{\text{span}}^{\text{cvx}}$. Moreover, we show that on a dense set, whose complement has zero Lebesgue measure, it does solve the equation. We state the result in terms of the failure to solve the corresponding HRMA (12), that corresponds to the HCMA on the open orbit M_o . Let

$$\Delta(\psi) := \{ (s, x) : \psi \text{ is finite and differentiable at } (s, x) \} \subset \mathbb{R}_+ \times \mathbb{R}^n,$$

denote the regular locus of ψ , and let

$$\Sigma_{\text{sing}} := \mathbb{R}_+ \times \mathbb{R}^n \setminus \Delta(\psi),$$

denote its singular locus. Since ψ is everywhere finite, the former is dense while the latter has Lebesgue measure zero in $\mathbb{R}_+ \times \mathbb{R}^n$. Set,

$$\Sigma_{\text{sing}}(T) := [0, T] \times \mathbb{R}^n \setminus \Delta(\psi).$$

THEOREM 2. (See [RZ2].) *(i) ψ solves the HRMA (12) on the dense regular locus,*

$$\text{MA}\psi = 0 \quad \text{on} \quad \Delta(\psi) \subset \mathbb{R}_+ \times \mathbb{R}^n.$$

In addition, $[0, T_{\text{span}}^{\text{cvx}}) \times \mathbb{R}^n \subset \Delta(\psi)$.

(ii) Whenever $T > T_{\text{span}}^{\text{cvx}}$, ψ fails to solve the HRMA (12). In particular, the Monge-Ampère measure of ψ charges the set $\Sigma_{\text{sing}}(T)$ with positive mass,

$$\int_{[0, T] \times \mathbb{R}^n} \text{MA}\psi = \int_{\Sigma_{\text{sing}}(T)} \text{MA}\psi > 0.$$

Equivalently, $\varphi = \varphi_\infty$ ceases to solve the HCMA (10), when $T > T_{\text{span}}^{\text{cvx}}$. However, it does solve the HCMA on a dense set in $S_T \times M$.

It is well-known that the Legendre transform linearizes the HRMA, and hence that the Legendre transform potential ψ is a solution as long as it is sufficiently smooth or equivalently as long as the symplectic potential is strictly convex. It does not seem to have been observed before that the Legendre transform potential fails to solve the HRMA as soon as it ceases to be differentiable. Theorems 1 and 2 come close to settling Conjecture 2.4 in the case of toric or Abelian varieties. They leave open the possibility that there exists an alternative method to solve the HRMA. That possibility is investigated in [RZ3], where it is shown that the Legendre solution is in a sense the optimal subsolution among several natural approaches.

In order to prove Theorem 1 we first show that the operators U_N quantize the Hamiltonian flow of $X_{\varphi_0}^{\omega_{\varphi_0}}$. This result holds on any projective Kähler manifold and does not make use of symmetry. The proof is based on the Toeplitz calculus developed by Boutet de Monvel-Sjöstrand [BSj] and Boutet de Monvel-Guillemin [BG]. We then show that U_N is well approximated by a second type of quantization that uses the symplectic potential.

The analysis of the logarithmic asymptotics of $U_N(-\sqrt{-1}s, z, z)$ is closely related to the analysis of families of toric Bergman metrics in [SoZ1, Z4], and these techniques allow us to compute the asymptotic spectrum of these operators and conclude the C^2 convergence up to $T < T_{\text{span}}^{\text{cvx}}$. Finally, we prove the global C^0 convergence to the Legendre transform subsolution. The logarithmic classical limit is closer to large deviations theory than to semi-classical Toeplitz analysis since it involves the analytic continuation in time of the Toeplitz quantization and not the quantization itself.

3.1. Further results. As mentioned above, we prove in [RZ2] that the Legendre transform potential fails to solve the equation even in a weak sense after the convex lifespan. Consequently the quantization method fails to solve the equation after this time, at least in the case of the HRMA.

But it is plausible that the quantization method produces the solution as long as a weak solution exists, and that it is in some sense the “optimal” sub-solution. To prove this, it is necessary to investigate whether there exist other ways of solving the Cauchy problem after the convex lifespan. This is initiated in a subsequent article [RZ3] in the series where we characterize the smooth lifespan of the more general HCMA in terms of analytic continuation of Hamiltonian dynamics. In the case of the HRMA this characterization shows precisely that $T_{\text{span}}^{\infty} = T_{\text{span}}^{\text{cvx}}$, and hence no smooth solution exist beyond the convex lifespan. By Theorem 1 this shows that the quantization approach solves the Cauchy problem for as long as a smooth solution exists.

We also introduce the notion of a leafwise subsolution, and show that the Legendre transform potential is the unique leafwise subsolution to the Cauchy problem. Also, in a further sequel we show that in a certain class of admissible subsolutions it is impossible to solve the Cauchy problem for the HRMA beyond the convex lifespan. This comes sufficiently close to confirming Conjecture 2.4 in the cases of toric Kähler manifolds and Abelian varieties with Cauchy data invariant under $(S^1)^n$.

Among Kähler manifolds without large symmetry, it seems most feasible to study the Cauchy problem for HCMA on a Riemann surface. The results and methods of this series suggest a general conjecture on the lifespan of solutions in that case. We plan to discuss it elsewhere.

4. BACKGROUND

4.1. Kähler quantization. Our setting consists of a Kähler manifold (M, ω) of complex dimension n with $[\omega] \in H^2(M, \mathbb{Z})$. Under this integrality condition, there exists a positive Hermitian holomorphic line bundle $(L, h) \rightarrow M$ whose curvature form is given locally by

$$\omega \equiv \omega_h = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \|e_L\|_h,$$

where e_L is a nonvanishing local holomorphic section of L , and where $\|e_L\|_h = h(e_L, e_L)^{1/2}$ denotes the h -norm of e_L .

The Hilbert spaces ‘quantizing’ (M, ω) are then defined to be the spaces

$$H^0(M, L^N)$$

of holomorphic sections of $L^N = L \otimes \cdots \otimes L$. The metric h induces Hermitian metrics h^N on L^N given by $\|s^{\otimes N}\|_{h^N} = \|s\|_h^N$. We give $L^2(M, L^N)$ the inner product

$$\|s\|_{\text{Hilb}_N(h)}^2 := \frac{1}{V} \int_M |s|_{h^N}^2 (N\omega_h)^n. \quad (16)$$

We then define the Szegő kernels as the Schwartz kernels $\Pi_N(z, w)$ of the orthogonal projections $\Pi_N : L^2(M, L^N) \rightarrow H^0(M, L^N)$ with respect to this inner product, so that

$$(\Pi_N s)(y) = \int_M \Pi_N(x, y) s(x) (N\omega(x))^n, \quad s \in L^2(M, L^N). \quad (17)$$

(Note that Π_N depends on h although we omit that from the notation.)

Instead of dealing with sequences of Hilbert spaces, observables and unitary operators on M , it is convenient to lift them to the circle bundle

$$X = \{\lambda \in L^* : \|\lambda\|_{h^{-1}} = 1\},$$

where L^* is the dual line bundle to L , and where h^{-1} is the norm on L^* dual to h . Let us now describe the lifted objects.

Let ρ be the function $\|\lambda\|_{h^{-1}} - 1$ on L^* . Associated to X is the contact form $\alpha = -\sqrt{-1} \partial \rho|_X = \sqrt{-1} \bar{\partial} \rho|_X$ and the volume form

$$(d\alpha)^n \wedge \alpha = \pi^* \omega^n \wedge \alpha. \quad (18)$$

We let $r_\theta w = e^{\sqrt{-1}\theta} w$, $w \in X$, denote the S^1 action on X and denote its infinitesimal generator by $\frac{1}{\sqrt{-1}} \frac{\partial}{\partial \theta}$. Holomorphic sections then lift to elements of the Hardy space $H^2(X) \subset L^2(X)$ of square-integrable CR functions on X , i.e., functions that are annihilated by the Cauchy-Riemann operator $\bar{\partial}_b := \pi^{0,1} \circ d$ (where $TX \otimes_{\mathbb{R}} \mathbb{C} =$

$T^{1,0}X \oplus T^{0,1}X \oplus \mathbb{C}\frac{\partial}{\partial\theta}$ and $\pi^{0,1}$ is defined as the projection onto the second factor) and are L^2 with respect to the inner product

$$\langle F_1, F_2 \rangle = \frac{1}{2\pi V} \int_X F_1 \overline{F_2} (d\alpha)^n \wedge \alpha, \quad F_1, F_2 \in L^2(X). \quad (19)$$

The S^1 action on X gives a representation of S^1 on $L^2(X)$ with irreducible pieces denoted $L_N^2(X)$. We thus have the Fourier decomposition,

$$L^2(X) = \bigoplus_{N \geq 0} L_N^2(X). \quad (20)$$

We denote by \mathbf{D} the operator on $L^2(X)$ with spectrum \mathbb{Z} and whose N -th eigenspace $L_N^2(X)$ consists of functions transforming by $e^{\sqrt{-1}N\theta}$ under the S^1 action r_θ on X . Thus,

$$\mathbf{D} = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial\theta}, \quad (21)$$

the infinitesimal generator of the S^1 action.

Since the S^1 action on X commutes with $\bar{\partial}_b$ we also have $H^2(X) = \bigoplus_{N=0}^{\infty} H_N^2(X)$ where

$$H_N^2(X) := \{F \in H^2(X) : F(r_\theta w) = e^{\sqrt{-1}N\theta} F(w)\} = L_N^2(X) \cap \ker \bar{\partial}_b.$$

A section s_N of L^N determines an equivariant function \hat{s}_N on L^* by the rule

$$\hat{s}_N(\lambda) = (\lambda^{\otimes N}, s_N(z)), \quad \lambda \in L_z^*, \quad z \in M, \quad (22)$$

where $\lambda^{\otimes N} = \lambda \otimes \cdots \otimes \lambda$. We henceforth restrict \hat{s} to X and then the equivariance property takes the form $\hat{s}_N(r_\theta w) = e^{iN\theta} \hat{s}_N(w)$. Up to a factor of N^n the map $s \mapsto \hat{s}$ is a unitary equivalence between $H^0(M, L^N)$ and $H_N^2(X)$.

We now define the (lifted) Szegő kernel of degree N to be the Schwartz kernel of the orthogonal projection $\tilde{\Pi}_N : L^2(X) \rightarrow H_N^2(X)$. It is defined by

$$\tilde{\Pi}_N F(w) = \frac{1}{2\pi V} \int_X \tilde{\Pi}_N(w, v) F(v) (d\alpha)^n \wedge \alpha(v), \quad F \in L^2(X). \quad (23)$$

The full Szegő kernel is then

$$\tilde{\Pi} = \sum_{N=1}^{\infty} \tilde{\Pi}_N. \quad (24)$$

To simplify notation we will from now on omit the tilde from the lifted projection operators on X and simply write Π, Π_N .

It was proved by Boutet de Monvel and Sjostrand [BSj] (see also the Appendix to [BG]) that Π is a complex Fourier integral operator (FIO) of positive type,

$$\Pi \in I_c^0(X \times X, \mathcal{C}) \quad (25)$$

associated to a positive canonical relation \mathcal{C} . For definitions and notation concerning complex FIO we refer to [MS, BSj, BG]. The real points of \mathcal{C} form the diagonal $\Delta_{\Sigma \times \Sigma}$ in the square of the symplectic cone

$$\Sigma := \{(w, r\alpha(w)) : r > 0, w \in X\} \subset T^*X, \quad (26)$$

where α is the connection, or contact, form. We refer to [BG], Appendix, Lemma 4.5. Let ω_{T^*X} denote the canonical symplectic form on T^*X , and let

$$\omega_\Sigma := \omega_{T^*X}|_\Sigma \quad (27)$$

denote its restriction to Σ , a symplectic form on Σ .

Finally, recall that a Toeplitz operator is an operator of the form $\Pi A \Pi$ where A is a pseudo-differential operator, and a (complex) Toeplitz Fourier integral operator is one where A is allowed to be a (complex) Fourier integral operator. When A is a pseudo-differential operator we denote by s_A its full symbol, and by σ_A its principal symbol. If B is a (complex) Fourier integral operator we denote by σ_B its symbol. Lastly, the symbol of $\Pi B \Pi$ is given by $\sigma_A|_\Sigma$ [BSj].

4.2. Toric Kähler manifolds. We now review some geometry and analysis on toric Kähler manifolds. Fuller details and exposition can be found in [A, G, R, RZ1, SoZ1, STZ].

Let $\mathbf{T} := (S^1)^n$. A symplectic toric manifold is a compact closed Kähler manifold (M, ω) whose automorphism group contains a complex torus $(\mathbb{C}^*)^n$ whose action on a generic point is an open dense orbit isomorphic to $(\mathbb{C}^*)^n$, and for which the real torus $\mathbf{T} \subset (\mathbb{C}^*)^n$ acts in a Hamiltonian fashion by isometries.

We will work with coordinates on the open dense orbit

$$M_o \cong (\mathbb{C}^*)^n$$

of the complex torus given by

$$z = e^{x/2 + \sqrt{-1}\theta}, \quad (x, \theta) \in \mathbb{R}^n \times (S^1)^n. \quad (28)$$

Let $\omega|_{M_o} = \sqrt{-1}\partial\bar{\partial}\psi$. The work of Atiyah and Guillemin-Sternberg [At, GS2] implies that the image of the moment map $\nabla\psi$ is a convex polytope $P \subset \mathbb{R}^n$ and depends only on $[\omega]$. We further assume that this is a lattice polytope. Being a lattice Delzant polytope [De1] means that: (i) at each vertex meet exactly n edges, (ii) each edge is the set of points $\{p + tu_{p,j} : t \geq 0\}$ with $p \in \mathbb{Z}^n$ a vertex, $u_{p,j} \in \mathbb{Z}^n$ and $\text{span}\{u_{p,1}, \dots, u_{p,n}\} = \mathbb{Z}^n$. Equivalently, there exist outward pointing normal vectors $\{v_j\}_{j=1}^d \subset \mathbb{Z}^n$, with v_j normal to the j -th $(n-1)$ -dimensional face of P (also called a facet), that are primitive (i.e., their components have no common factor), and P may be written as

$$P = \{y \in \mathbb{R}^n : l_j(y) := \langle y, v_j \rangle - \lambda_j \geq 0, \quad j = 1, \dots, d\},$$

with $\lambda_j = \langle p, v_j \rangle \in \mathbb{Z}$ with p any vertex on the j -th facet, and y the coordinate on \mathbb{R}^n . Note that the main results in this article extend to orbifold toric varieties, since we only make essential use of (i).

The Kähler form ω is the curvature $(1,1)$ form of a line bundle $L \rightarrow M$. A basis for the space $H^0(M, L)$ of holomorphic sections is given by the monomials $\chi_\alpha(z) = z^\alpha$ with $\alpha \in P$. More generally, $H^0(M, L)$ generates the coordinate ring $\bigoplus_{N=1}^\infty H^0(M, L^N)$, and each lattice point γ in NP corresponds to a section χ_γ of $L^N \rightarrow M$ defined by

$$\chi_\gamma = \chi_{\beta_1} \otimes \dots \otimes \chi_{\beta_N}, \quad (29)$$

where $\beta_1, \dots, \beta_N \in P$ such that $\gamma = \beta_1 + \dots + \beta_N$ (see [STZ]).

We now consider the homogenization (lift to X) of toric Kähler manifolds. The lattice points in NP for each $N \in \mathbb{N}$ correspond in X to the ‘homogenized’ lattice points $\widehat{NP} \subset \mathbb{Z}^{n+1}$ of the form

$$\widehat{\alpha}^N = \widehat{\alpha} := (\alpha_1, \dots, \alpha_n, Np - |\alpha|), \quad \alpha = (\alpha_1, \dots, \alpha_n) \in NP \cap \mathbb{Z}^n,$$

where $p = \max_{\beta \in P \cap \mathbb{Z}^n} |\beta|$. For simplicity, we generally assume henceforth that $p = 1$. We also define the cone

$$\Lambda_P := \bigcup_{N=1}^{\infty} \widehat{NP}.$$

Rays $\mathbb{N}\widehat{\alpha}$ in this cone define the semiclassical limit.

The monomials χ_α lift to the CR monomials $\widehat{\chi}_{\widehat{\alpha}}(w) \equiv \widehat{\chi}_\alpha(w)$, $w \in X$ (see (22)), for $\widehat{\alpha} \in \Lambda_P$. They are joint eigenfunctions of a quantized torus action on X . Let

$$\xi_j := \frac{\partial}{\partial \theta_j}, \quad 1 \leq j \leq n,$$

denote the Hamiltonian vector fields generating the \mathbf{T} action on M . We use the connection form α to define the horizontal lifts ξ_j^h of the Hamiltonian vector fields ξ_j :

$$\pi_* \xi_j^h = \xi_j, \quad \alpha(\xi_j^h) = 0, \quad 1 \leq j \leq n. \quad (30)$$

Let $\xi_j^* \in \mathbb{R}^n$ denote the element of the Lie algebra of \mathbf{T} which acts as ξ_j on M . We then define the vector fields Ξ_j by:

$$\Xi_j := \xi_j^h + 2\pi\sqrt{-1}\langle \nabla\psi \circ \pi, \xi_j^* \rangle \frac{\partial}{\partial \theta} = \xi_j^h + 2\pi\sqrt{-1}(\nabla\psi \circ \pi)_j \frac{\partial}{\partial \theta}, \quad 1 \leq j \leq n. \quad (31)$$

Finally, we define the differential operators (lifted action operators),

$$\hat{I}_j := \Xi_j, \quad j = 1, \dots, n, \quad \hat{I}_{n+1} := \frac{1}{\sqrt{-1}} \frac{\partial}{\partial \theta} - \sum_{j=1}^n \Xi_j. \quad (32)$$

We recall that $\frac{1}{\sqrt{-1}} \frac{\partial}{\partial \theta}$ is abbreviated by \mathbf{D} and note that \hat{I}_{n+1} is not the same as \mathbf{D} . Then the monomials $\widehat{\chi}_{\widehat{\alpha}}$ are the joint CR eigenfunctions of $(\hat{I}_1, \dots, \hat{I}_{n+1})$ for the joint eigenvalues $\widehat{\alpha} \in \Lambda_P$, i.e.,

$$\hat{I}_j \widehat{\chi}_{\widehat{\alpha}} = \widehat{\alpha}_j \widehat{\chi}_{\widehat{\alpha}}, \quad \widehat{\alpha} \in \Lambda_P, \quad \bar{\partial}_b \widehat{\chi}_{\widehat{\alpha}} = 0, \quad j = 1, \dots, n+1. \quad (33)$$

For simplicity of notation, we denote by $D_{\hat{I}}$ the vector of first-order operators

$$D_{\hat{I}} := \frac{1}{2\pi\sqrt{-1}} (\hat{I}_1, \dots, \hat{I}_n), \quad (34)$$

and use the same notation for the quantized torus action on $H^0(M, L^N)$ and on X .

Although we are primarily concerned with holomorphic sections over M and their lifts as CR holomorphic functions on X , we need to consider non-CR holomorphic eigenfunctions of the action operators as well. We thus need to consider the anti-Hardy space $\overline{\mathcal{H}}^2(X)$ of anti-CR functions, i.e. solutions of $\partial_b f = 0$. A Hilbert basis is given by the complex-conjugate monomials $\overline{\widehat{\chi}_{\widehat{\alpha}}}$.

Products of eigenfunctions are also eigenfunctions. Hence, the orthonormal mixed monomials

$$\hat{\chi}_{\hat{\alpha}, \hat{\beta}}(x) = \hat{\chi}_{\hat{\alpha}} \overline{\hat{\chi}_{\hat{\beta}}}$$

are eigenfunctions of eigenvalue $\hat{\alpha} - \hat{\beta}$ for $\{\hat{I}_1, \dots, \hat{I}_{n+1}\}$. It can be shown [STZ] that

$$L^2(X) = \bigoplus_{\hat{\alpha}, \hat{\beta} \in \Lambda_P} \mathbb{C} \hat{\chi}_{\hat{\alpha}, \hat{\beta}}. \quad (35)$$

It follows that the joint spectrum of $(\hat{I}_1, \dots, \hat{I}_{n+1})$ on $L^2(X)$ is given by

$$\text{Spec}(\hat{I}_1, \dots, \hat{I}_{n+1}) = \Lambda_P - \Lambda_P = \mathbb{Z}^{n+1}. \quad (36)$$

4.3. Convex analysis. Here we define some basic notation related to convex functions. For general background on Legendre duality and convexity we refer the reader to [Ro].

A vector $v \in (\mathbb{R}^n)^*$ is said to be a subgradient of a function f at a point x if $f(z) \geq f(x) + \langle v, z - x \rangle$ for all z . The set of all subgradients of f at x is called the subdifferential of f at x , denoted $\partial f(x)$.

The Legendre-Fenchel conjugate of a continuous function $f = f(x)$ on \mathbb{R}^n is defined by

$$f^*(y) := \sup_{x \in \mathbb{R}^n} (\langle x, y \rangle - f(x)).$$

For simplicity, we will refer to f^* sometimes as the Legendre dual, or just dual, of f . An open-orbit Kähler potential $\psi \in \mathcal{H}(\mathbf{T})$ is a smooth strictly convex function on \mathbb{R}^n in logarithmic coordinates. Therefore its gradient $\nabla \psi$ is one-to-one onto $P = \text{Im} \nabla \psi$ and one has the following explicit expression for its Legendre dual ([Ro], or [R], p. 84–87),

$$u(y) = \psi^*(y) = \langle y, (\nabla \psi)^{-1}(y) \rangle - \psi \circ (\nabla \psi)^{-1}(y), \quad (37)$$

which is a smooth strictly convex function on P , satisfying

$$\nabla u(y) = (\nabla \psi)^{-1}(y). \quad (38)$$

Following Guillemin [G], the function u is called the symplectic potential of $\sqrt{-1}\partial\bar{\partial}\psi$. The space of all symplectic potentials is denoted by $\mathcal{LH}(\mathbf{T})$. Put

$$u_G := \sum_{k=1}^d l_k \log l_k. \quad (39)$$

A result of Guillemin [G] states that for any symplectic potential u the difference $u - u_G$ is a smooth function on P (that is, up to the boundary). In other words, (13) may be rewritten as

$$\mathcal{LH}(\mathbf{T}) = \{u \in C^\infty(P \setminus \partial P) : u = u_G + F, \text{ with } F \in C^\infty(P)\}. \quad (40)$$

5. QUANTIZING THE HAMILTONIAN FLOW OF $\dot{\varphi}_0$

In this section (M, ω) is an arbitrary projective Kähler manifold. The first step in defining the analytic continuation of $\exp tX_{\dot{\varphi}_0}$ is to quantize this Hamiltonian flow. We use the method of Toeplitz quantization [BG, Z2] (see also [Z1], §5, for some exposition). We may state the result either in terms of one homogeneous Fourier integral operator on $L^2(X)$ or as a semi-classical Fourier integral operator on each of the spaces $L_N^2(X)$ in the decomposition (20).

It should be noted that the quantization we use is not unique, i.e., there exists more than one unitary group of Toeplitz Fourier integral operators with underlying canonical flow equal to the Hamiltonian flow of $\dot{\varphi}_0$. Indeed, for any unitary pseudo-differential operator $V = e^{\sqrt{-1}A}$ obtained by exponentiating a self-adjoint pseudo-differential operator A of degree zero, and any quantization $U(t)$ of $\exp tX_{\dot{\varphi}_0}$, the operator $V^*U(t)V$ is another quantization with the same principal symbol. This lack of uniqueness will be seen below in the fact that we have more than one version of the quantization. They are closely related and differ by lower order terms.

To quantize the classical Hamiltonian, we first quantize the Hamiltonian as the zeroth order Toeplitz operator $\Pi\dot{\varphi}_0\Pi$ on $H^2(X)$ where $\dot{\varphi}_0$ denotes the multiplication operator by $\dot{\varphi}_0$. It is a bounded Hermitian Toeplitz operator.

DEFINITION 5.1. *Define the one-parameter subgroup $U(t)$ of unitary operators on $L^2(X)$ by (cf. (21))*

$$U(t) = \Pi e^{\sqrt{-1}t\Pi\dot{\varphi}_0\Pi}\Pi. \quad (41)$$

Its Fourier components are given by

$$U_N(t) = \Pi_N e^{\sqrt{-1}tN\Pi_N\dot{\varphi}_0\Pi_N}\Pi_N.$$

We note that $U(t)$ is not quite the same as $\Pi e^{\sqrt{-1}tD\dot{\varphi}_0}\Pi$, which is manifestly the composition of complex Fourier integral operators. However, $\Pi_N\dot{\varphi}_0\Pi_N$ is the quantization of $\dot{\varphi}_0$. We compose $e^{\sqrt{-1}tN\Pi_N\dot{\varphi}_0\Pi_N}$ with Π_N to make the operator preserve $H^0(M, L^N)$. Note that $U(t) = \Pi e^{\sqrt{-1}t\Pi\dot{\varphi}_0\Pi}\Pi = e^{\sqrt{-1}t\Pi D\dot{\varphi}_0}\Pi$.

We now verify that $U(t)$ is a complex Fourier integral operator with underlying canonical relation equal to graph of the Hamiltonian flow at time t of $r\dot{\varphi}_0$ on (Σ, ω_Σ) , where r and (Σ, ω_Σ) are defined in (26)–(27). This is the content of saying that $U_N(t)$ is a quantization of the Hamiltonian flow of $\dot{\varphi}_0$ on (M, ω_{φ_0}) .

PROPOSITION 5.2. *$U(t)$ is a group of complex Toeplitz Fourier integral operators on $L^2(X)$ whose underlying canonical relation is the graph of the time t Hamiltonian flow of $r\dot{\varphi}_0$ on the symplectic cone (Σ, ω_Σ) .*

Proof. We first observe that $U(t)$ is characterized as the unique solution of the ordinary differential equation

$$\frac{d}{dt}U(t) = (\sqrt{-1}\Pi D\dot{\varphi}_0\Pi)U(t), \quad U(0) = \Pi.$$

We use the following result of Boutet de Monvel-Guillemin, whose proof we sketch later.

LEMMA 5.3. (see [BG], Proposition 2.13) *Let T be a Toeplitz operator on Σ of order p . Then there exists a pseudo-differential operator Q of order p on X such that $[Q, \Pi] = 0$ and $T = \Pi Q \Pi$.*

We apply Lemma 5.3 to $T = \Pi \dot{\varphi}_0 \Pi$. Thus, there exists a zeroth order pseudo-differential operator Q on X with $\sigma_Q|_\Sigma = \dot{\varphi}_0|_\Sigma$ (see [BG], Theorem 2.9 and Proposition 2.13 for background). Note that here we identify $\dot{\varphi}_0$ with its lift to $\Sigma \subset T^*X$.

Since $\Pi e^{\sqrt{-1}t\Pi\mathbf{D}Q\Pi}\Pi$ and $\Pi e^{\sqrt{-1}t\mathbf{D}Q}\Pi$ satisfy the same differential equation

$$\frac{d}{dt}W(t) = \sqrt{-1}\Pi\mathbf{D}Q\Pi W(t)$$

and have the same initial condition, we have

$$U(t) = \Pi e^{\sqrt{-1}t\Pi\mathbf{D}Q\Pi}\Pi = \Pi e^{\sqrt{-1}t\mathbf{D}Q}\Pi. \quad (42)$$

Here, we use that $\Pi^2 = \Pi$ hence $\Pi Q = \Pi Q \Pi$ and that Π and \mathbf{D} commute.

Now $e^{\sqrt{-1}t\mathbf{D}Q}$ is the exponential of a real principal type pseudo-differential operator of order one on $L^2(X)$ and hence is a unitary group of Fourier integral operators on $L^2(X)$ quantizing the Hamiltonian flow of $\sigma_{\mathbf{D}Q}$ on T^*X . Since Π is a complex Fourier integral operator whose real canonical relation is the diagonal in $\Sigma \times \Sigma$ [BSj], $U(t)$ is also a complex Fourier integral operator. To complete the proof of the Proposition, it suffices to prove that the canonical relation of $U(t)$ is the graph of the time t Hamiltonian flow of $r\dot{\varphi}_0$ on (Σ, ω_Σ) .

Let Ψ_t denote the time t Hamiltonian flow of $\sigma_{\mathbf{D}Q}$ on (T^*X, ω_{T^*X}) . By the composition theorem for complex Fourier integral operators [MS], the operator $\Pi e^{\sqrt{-1}t\mathbf{D}Q}\Pi$ is a complex Fourier integral operator whose canonical relation is the set-theoretic composition

$$\begin{aligned} & \{(v, v) : v \in \Sigma\} \circ \{(p, \Psi_t(p)) : p \in T^*X\} \circ \{(q, q) : q \in \Sigma\} \\ &= \{(m, \Psi_t(m)) : m \in \Sigma\} \cap \Sigma \times \Sigma. \end{aligned} \quad (43)$$

Here we make use of the fact that the symbol of Π is nowhere vanishing on Σ and that of $e^{\sqrt{-1}t\mathbf{D}Q}$ is nowhere vanishing on the graph of Ψ_t . It only remains to equate (43) with the graph of the time t Hamiltonian flow of $r\dot{\varphi}_0$ on (Σ, ω_Σ) .

Since $[\Pi, Q] = 0$, we have

$$\Pi e^{\sqrt{-1}t\mathbf{D}Q} = \Pi e^{\sqrt{-1}t\mathbf{D}Q}\Pi.$$

This implies that the canonical relations of both sides in this equation must be equal. The canonical relation of the left hand side equals

$$\{(v, v) : v \in \Sigma\} \circ \{(p, \Psi_t(p)) : p \in T^*X\} = \{(q, \Psi_t(q)) : q \in \Sigma\}. \quad (44)$$

Equating this to (43) it follows that Ψ_t preserves Σ . Hence, the Hamiltonian vector field $X_{\sigma_{\mathbf{D}Q}}^{T^*X}$ of $\sigma_{\mathbf{D}Q}$ with respect to ω_{T^*X} is tangent to the symplectic sub-cone Σ .

We note that the symbol of \mathbf{D} is the Clairaut integral $\sigma_{\mathbf{D}}(x, \xi) = \langle \xi, \frac{\partial}{\partial \theta} \rangle$. Since $\alpha(\frac{\partial}{\partial \theta}) = 1$ (see, e.g., [R], p. 69), it follows from (26) that $\sigma_{\mathbf{D}}|_\Sigma = r$. Recall also that

$\sigma_Q|_\Sigma = \dot{\varphi}_0|_\Sigma$. Thus, to complete the proof it remains to show that the restriction of Ψ_t to Σ is the Hamiltonian flow of

$$\sigma_{\mathbf{D}}\sigma_Q|_\Sigma = r\dot{\varphi}_0 \quad (45)$$

on (Σ, ω_Σ) . Let $X_{r\dot{\varphi}_0}^\Sigma$ be the Hamiltonian vector field of $\sigma_{\mathbf{D}}\sigma_Q|_\Sigma$ with respect to ω_Σ . At a point of Σ , we have

$$\omega_{T^*X}(X_{\sigma_{\mathbf{D}}\sigma_Q}^{T^*X}, \cdot) = d\sigma_{\mathbf{D}}\sigma_Q, \quad \omega_\Sigma(X_{r\dot{\varphi}_0}^\Sigma, \cdot) = d(r\dot{\varphi}_0).$$

Evaluating these 1-forms on all tangent vectors $Y \in T\Sigma$, and using (27), (45), and that $X_{\sigma_{\mathbf{D}}\sigma_Q}$ is tangent to Σ , we conclude that $X_{r\dot{\varphi}_0}^\Sigma = X_{\sigma_{\mathbf{D}}\sigma_Q}^{T^*X}$. This completes the proof of Proposition 5.2. \square

REMARK 5.4. The fact that the Hamiltonian vector field $X_{\sigma_{\mathbf{D}}\sigma_Q}$ of $\sigma_{\mathbf{D}}\sigma_Q$ preserves Σ is equivalent to the fact that the Hamilton vector field X_{σ_Q} preserves Σ . Indeed,

$$X_{\sigma_{\mathbf{D}}\sigma_Q} = \sigma_{\mathbf{D}}X_{\sigma_Q} + \sigma_QX_{\sigma_{\mathbf{D}}}.$$

Since $[\mathbf{D}, Q] = 0$ and hence $\{\sigma_{\mathbf{D}}, \sigma_Q\}_{\omega_{T^*X}} = 0$ the flows of X_{σ_Q} and of $X_{\sigma_{\mathbf{D}}}$ commute. Hence the Hamiltonian flow of $X_{\sigma_{\mathbf{D}}\sigma_Q}$ equals the composition

$$\exp t\sigma_{\mathbf{D}}X_{\sigma_Q} \circ \exp t\sigma_QX_{\sigma_{\mathbf{D}}}.$$

The restricted vector field $X_{\sigma_{\mathbf{D}}}|_\Sigma = \frac{\partial}{\partial \theta}|_\Sigma$ is equal to $X_{\sigma_{\mathbf{D}}|_\Sigma}$ since the principal S^1 -action preserves Σ (by (26), as it preserves α). Hence its flow always preserves Σ . The fact that the flow of X_{σ_Q} preserves Σ is proved in [BG], Proposition 11.4 and the Remark following it. The proof uses the construction of Q and Toeplitz symbol calculus, and is therefore similar to the one given above.

For the sake of completeness, we briefly sketch a proof of Lemma 5.3, following the proof of a similar assertion in [GS1], Theorem 5.8 and Lemma 5.9.

LEMMA 5.5. *Given a smooth real-valued function q on M , homogeneous of degree zero, there exists a self-adjoint pseudo-differential operator Q such that $[Q, \Pi] = 0$ and such that $\sigma_Q|_\Sigma = q$.*

The proof uses symbol calculus and spectral theory, all of which are available in the Toeplitz setting. The first observation is that the principal symbol of $[\Pi, \dot{\varphi}_0]$ vanishes, hence it is complex Fourier integral operator (or more specifically Toeplitz operator) of order -1 . By adding an operator Q_{-1} to $\dot{\varphi}_0$ and using transport equations for the symbol, one can arrange that the symbols of order -1 and order -2 of $[\Pi, \dot{\varphi}_0 + Q_{-1}]$ equal zero. By repeating infinitely often and asymptotically summing the operators, one can find \tilde{Q} such that $[\Pi, \tilde{Q}] = 0$ and $\Pi\dot{\varphi}_0\Pi - \Pi\tilde{Q}\Pi$ are smoothing. One then puts $Q = \tilde{Q} + \Pi\dot{\varphi}_0\Pi - \Pi\tilde{Q}\Pi$.

6. TWO QUANTIZATIONS OF THE HAMILTONIAN FLOW ON A TORIC MANIFOLD

In this section we specialize the construction of Section 5 from a general projective manifold to a toric manifold and study its asymptotic spectrum. We then give an alternative quantization of the Hamiltonian flow of $\dot{\varphi}_0$ in the special case of a toric manifold and compare the two quantizations. These results will then be applied in Section 7 to complete the proof of Theorem 1.

Recall from §§4.2 that the toric monomials $\{\chi_\alpha(z) := z^\alpha\}_{\alpha \in NP \cap \mathbb{Z}^n}$ are an orthogonal basis of $H^0(M, L^N)$ with respect to any toric-induced Hilbert space structure on this vector space. Hence any such toric inner product is completely determined by the L^2 norms (up to N^n/V), or “norming constants,” of the toric monomials—

$$\mathcal{Q}_{h^N}(\alpha) := \|\chi_\alpha\|_{h^N}^2 = \int_{(\mathbb{C}^*)^n} |z^\alpha|^2 e^{-N\psi} \omega_h^n. \quad (46)$$

Here we let $h = e^{-\psi}$ with $\psi \in \mathcal{H}(\mathbf{T})$. As in [SoZ1], we put

$$\mathcal{P}_{h^N}(\alpha, z) := \frac{|\chi_\alpha(z)|_{h^N}^2}{\|\chi_\alpha\|_{h^N}^2}.$$

6.1. The quantization of the Hamiltonian flow using the Kähler velocity.

In this subsection we study the one-parameter subgroup $U(t)$ given by Definition 5.1 on a toric manifold.

The first observation is that since $\dot{\varphi}_0$ is torus-invariant the multiplication operator $\dot{\varphi}_0$ preserves the block decomposition (20). Therefore the toric monomials diagonalize the Toeplitz operators $\Pi_N \dot{\varphi}_0 \Pi_N$, that is,

$$\Pi_N \dot{\varphi}_0 \Pi_N \chi_\alpha = \mu_{N,\alpha} \chi_\alpha, \quad (47)$$

for some real numbers $\{\mu_{N,\alpha}\}_{\alpha \in NP \cap \mathbb{Z}^n}$. Since $\{\chi_\alpha\}_{\alpha \in NP \cap \mathbb{Z}^n}$ are orthogonal with respect to a toric inner product we have

$$\mu_{N,\alpha} = \frac{1}{\mathcal{Q}_{h_0^N}(\alpha)} \int_M \dot{\varphi}_0 |\chi_\alpha|_{h_0^N}^2 \omega_{\varphi_0}^n. \quad (48)$$

Hence we have the the following expression for the level N quantum analytic continuation potential induced by $U(\sqrt{-1}s)$:

$$\varphi_N(s, z) = \frac{1}{N} \log U_N(-\sqrt{-1}s, z, z) = \frac{1}{N} \log \sum_{\alpha \in NP \cap \mathbb{Z}^n} e^{sN\mu_{N,\alpha}} \frac{|\chi_\alpha(z)|_{h_0^N}^2}{\mathcal{Q}_{h_0^N}(\alpha)}. \quad (49)$$

6.2. An alternative quantization using the symplectic potential. We now introduce a second quantization in the special case of a toric manifold for which the eigenvalues are special values of the velocity of the symplectic potential. In effect, it is an explicit construction of the operator Q in Lemma 5.3, at least to leading order (which is sufficient for our purposes).

DEFINITION 6.1. *Define the one-parameter subgroup $V(t)$ of unitary operators on $L^2(X)$ by*

$$V(t) = \Pi e^{-\sqrt{-1}t \mathbf{D} \dot{u}_0(D_{\hat{f}} \mathbf{D}^{-1})} \Pi.$$

Its Fourier components are given by

$$V_N(t) = \Pi_N e^{-\sqrt{-1}t N \dot{u}_0(N^{-1} D_{\hat{f}})} \Pi_N.$$

It follows that the level N quantum analytic continuation potential induced by $V(\sqrt{-1}s)$ is given by

$$\begin{aligned} \tilde{\varphi}_N(s, z) &:= \frac{1}{N} \log V_N(-\sqrt{-1}s, z, z) \\ &= \frac{1}{N} \log \sum_{\alpha \in NP \cap \mathbb{Z}^n} e^{-s N \dot{u}_0(\alpha/N)} \frac{|\chi_\alpha(z)|_{h_0^N}^2}{\mathcal{Q}_{h_0^N}(\alpha)}. \end{aligned} \tag{50}$$

In order to relate $\tilde{\varphi}_N$ to the actual quantum analytic continuation potentials φ_N the following fact is crucial.

PROPOSITION 6.2. *The sequence of unitary operators $\{V_N(t)\}_{N \geq 1}$ is a semi-classical complex Toeplitz Fourier integral operator quantizing the time t Hamiltonian flow of $\dot{\varphi}_0$ on (M, ω_{φ_0}) .*

We note that, equivalently, Proposition 6.2 could be stated in ‘homogeneous’ notation, that is, in an identical manner to Proposition 5.2 with $U(t)$ replaced by $V(t)$.

Proof. It is convenient to lift to the circle bundle X and use the full spectral theory of the action operators of §§4.2.

We observe that $\Pi \dot{u}_0(D_{\hat{f}} \mathbf{D}^{-1}) \Pi$ is defined by the Spectral Theorem to be the operator on

$$H^2(X) \setminus \mathbb{C} = \bigoplus_{N \in \mathbb{N}} H_N^2(X)$$

whose eigenfunctions are the same as the joint eigenfunctions of the quantum torus action, i.e., the lifted monomials

$$\{\hat{\chi}_{\hat{\alpha}} : \hat{\alpha} \in \Lambda_P\},$$

and whose corresponding eigenvalues are

$$\{\dot{u}_0(\alpha/N) : N \in \mathbb{N}, \alpha \in NP \cap \mathbb{Z}^n\}.$$

However, in order to apply classical results concerning operators of the form $e^{\sqrt{-1}tP}$ where P is a real first-order pseudo-differential operator of principal type we need to replace $\dot{u}_0(D_{\hat{f}} \mathbf{D}^{-1})$ with an operator defined on all of $L^2(X)$. Yet, since eventually we pre- and post-compose with Π , we are ultimately only interested in the restriction to $H^2(X) \setminus \mathbb{C}$ of the extended operator. Hence we would like the extended operator to coincide with $\dot{u}_0(D_{\hat{f}} \mathbf{D}^{-1})$ on $H^2(X) \setminus \mathbb{C}$. This is the purpose of the following Lemma.

LEMMA 6.3. *There exists a pseudo-differential operator R of order zero on $L^2(X)$ such that*

$$R|_{H^2(X) \setminus \mathbb{C}} = u_0(D_{\hat{f}} \mathbf{D}^{-1})|_{H^2(X) \setminus \mathbb{C}}. \quad (51)$$

Proof. There are two obstacles to defining $u_0(D_{\hat{f}} \mathbf{D}^{-1})$ on all of $L^2(X)$. First, according to (35)–(36) we need to define u_0 on \mathbb{R}^n , while originally it is only defined on P . Second, the operator \mathbf{D}^{-1} is only defined on the orthocomplement of the invariant functions on X for the S^1 action. The non-constant CR functions are orthogonal to the invariant functions, so $\Pi u_0(D_{\hat{f}} \mathbf{D}^{-1}) \Pi$ is well-defined on $H^2(X) \setminus \mathbb{C}$. But we wish to extend $u_0(D_{\hat{f}} \mathbf{D}^{-1})$ outside the Hardy space.

To deal with the first point, note that since u_0 is smooth up to the boundary of P , we may assume it is defined in some neighborhood of P in \mathbb{R}^n , and then multiply it by a smooth cutoff function η equal to 1 in a neighborhood of P and with compact support in \mathbb{R}^n . Then ηu_0 is a smooth function of compact support in \mathbb{R}^n , and $\eta u_0(D_{\hat{f}} \mathbf{D}^{-1}) \equiv (\eta u_0)(D_{\hat{f}} \mathbf{D}^{-1})$ is well-defined on $(\ker \mathbf{D})^\perp \subset L^2(X)$. As noted above,

$$\bigoplus_{N \in \mathbb{N}} H_N^2(X) = H^2(X) \setminus \mathbb{C} \subset (\ker \mathbf{D})^\perp,$$

and since

$$\text{Spec } D_{\hat{f}} \mathbf{D}^{-1}|_{H^2(X) \setminus \mathbb{C}} \subset P,$$

we have

$$\eta u_0(D_{\hat{f}} \mathbf{D}^{-1})|_{H^2(X) \setminus \mathbb{C}} = u_0(D_{\hat{f}} \mathbf{D}^{-1})|_{H^2(X) \setminus \mathbb{C}}. \quad (52)$$

We now turn to the second point. There are several ways of handling it; in addition to the construction that follows we mention two other possibilities in Remark 6.5 below. For any $\epsilon > 0$ let $\gamma_\epsilon = \gamma_\epsilon(\sigma_{\hat{f}}, \sigma_{\mathbf{D}}) \in C^\infty(T^*X \setminus \{0\})$ denote a homogeneous frequency cut-off, equal to 1 in an open conic neighborhood

$$\left\{ (x, \xi) \in T^*X \setminus \{0\} : |\sigma_{\mathbf{D}}| < \epsilon(|\sigma_{D_{\hat{f}}}|^2 + \sigma_{\mathbf{D}}^2)^{1/2} \right\} \quad (53)$$

of the set $\{\sigma_{\mathbf{D}} = 0\}$, and vanishing on

$$\left\{ (x, \xi) \in T^*X \setminus \{0\} : |\sigma_{\mathbf{D}}| > 2\epsilon(|\sigma_{D_{\hat{f}}}|^2 + \sigma_{\mathbf{D}}^2)^{1/2} \right\}$$

(note that $n+1$ of the vertical directions in T^*X are not involved). Let $\beta \in \mathbb{Z}^{n+1}$, and let $\chi_\beta \in L^2(X)$ be the associated monomial. Denote by $\gamma_\epsilon(D_{\hat{f}}, \mathbf{D})$ the Fourier multiplier associated to γ_ϵ , namely such that

$$\gamma_\epsilon(D_{\hat{f}}, \mathbf{D}) \chi_\beta(w) = \gamma_\epsilon(\beta) \chi_\beta(w).$$

This defines $\gamma_\epsilon(D_{\hat{f}}, \mathbf{D})$ on $L^2(X)$ (see (35)). Let I denote the identity operator on $L^2(X)$. Then $I - \gamma_\epsilon(D_{\hat{f}}, \mathbf{D})$ is a pseudo-differential operator with

$$\ker \mathbf{D} \subset \ker(I - \gamma_\epsilon(D_{\hat{f}}, \mathbf{D})),$$

and

$$R_\epsilon := \eta u_0(D_{\hat{f}} \mathbf{D}^{-1})(I - \gamma_\epsilon(D_{\hat{f}}, \mathbf{D}))$$

is a pseudo-differential operator of order zero, defined on all of $L^2(X)$.

To complete the proof of the Lemma, we will prove that (51) holds for $R := R_\epsilon$, for any $\epsilon > 0$ small enough.

Let $\hat{\alpha} \in \Lambda_P$ with $\chi_\alpha \in H^0(M, L^N)$, $\alpha \in NP \cap \mathbb{Z}^n$. We claim that for small enough $\epsilon > 0$ in (53) we have

$$\gamma_\epsilon(D_{\hat{f}}, \mathbf{D})\hat{\chi}_{\hat{\alpha}}(w) = \gamma_\epsilon(\alpha, N)\hat{\chi}_{\hat{\alpha}}(w) = 0.$$

For the second equality, note that for $(w, r\alpha(w)) \in \Sigma$, we have $\gamma_\epsilon(w, r\alpha(w)) = 0$ unless

$$r \leq 2\epsilon r(|\nabla\psi_0 \circ \pi(w)|^2 + 1)^{1/2}, \quad (54)$$

where $\pi : X \rightarrow M$ is the bundle projection map. For $r > 0$ equation (54) cannot hold if we take ϵ such that

$$0 < \epsilon < \frac{1}{2\sqrt{\sup_{y \in P} |y|^2 + 1}}, \quad (55)$$

since $\nabla\psi_0 \circ \pi(w) \in P$ (and P is a bounded set in \mathbb{R}^n). This proves the claim, for ϵ satisfying (55) (note that in the proof of the last assertion, instead of working in ‘homogeneous’ notation, we could have replaced $r > 0$ by $N \in \mathbb{N}$ and $r\nabla\psi_0 \circ \pi(w)$ by $\alpha \in NP$). It follows that

$$I = I - \gamma_\epsilon(D_{\hat{f}}, \mathbf{D}), \quad \text{on } H^2(X) \setminus \mathbb{C}.$$

Together with (52) this proves that

$$\dot{u}_0(D_{\hat{f}}\mathbf{D}^{-1})\Pi = \eta\dot{u}_0(D_{\hat{f}}\mathbf{D}^{-1})(I - \gamma_\epsilon(D_{\hat{f}}, \mathbf{D}))\Pi, \quad \text{for each } \epsilon \text{ satisfying (55),}$$

as desired. \square

The following Lemma is the concrete realization of Lemma 5.3 in the setting of toric Kähler manifolds.

LEMMA 6.4. *Let ϵ satisfy (55) and let $R := \eta\dot{u}_0(D_{\hat{f}}\mathbf{D}^{-1})(I - \gamma_\epsilon(D_{\hat{f}}, \mathbf{D}))$. The operator $\Pi R \Pi$ is a Toeplitz operator of order zero and its symbol is given by*

$$\sigma_R(w, \xi) = \dot{u}_0 \circ \nabla\psi_0 \circ \pi(w) = -\dot{\varphi}_0 \circ \pi(w), \quad (w, \xi) \in \Sigma,$$

where $\pi : X \rightarrow M$ is the projection onto the base.

Proof. As noted in the proof of Lemma 6.3, the symbol of $I - \gamma_\epsilon(D_{\hat{f}}, \mathbf{D})$ equals one on Σ . In addition, when restricting to Σ , the operator $\dot{u}_0(D_{\hat{f}}\mathbf{D}^{-1})$ has a well-defined symbol, equal to the symbol of $\eta\dot{u}_0(D_{\hat{f}}\mathbf{D}^{-1})$, restricted to Σ .

On Σ , the symbols of the vector fields ξ_j^h (see (30)) are the Clairaut integrals

$$\sigma_{\xi_j^h}(w, r\alpha(w)) = \alpha_w(\xi_j^h) = 0.$$

Hence, on Σ , the symbol of $\hat{I}_j, 1 \leq j \leq n$, is that of the second term in (31): $2\pi\sqrt{-1}r(\nabla\psi_0 \circ \pi)_j$. By the normalization of (34) therefore

$$\sigma_{D_{\hat{f}}}(w, r\alpha(w)) = r\nabla\psi_0 \circ \pi(w).$$

Since $\sigma_{\mathbf{D}^{-1}}(w, r\alpha(w)) = 1/r$ (see the proof of Proposition 5.2), it follows that the symbol of $\dot{u}_0(D_{\hat{f}}\mathbf{D}^{-1})$, restricted to Σ , is $\dot{u}_0(\pi^*\nabla\psi_0)$ and thus equals the stated

Hamiltonian σ_R . It is the lift of the Hamiltonian $H(z) = \dot{u}_0 \circ \nabla \psi_0(z)$ to the cone $\Sigma = \Sigma_{h_0}$. \square

We may now conclude the proof of Proposition 6.2. Indeed, from Lemma 6.3 we have that

$$V(t) = \Pi e^{-\sqrt{-1}t\mathbf{D}R}\Pi.$$

Since $\mathbf{D}R$ is a real principal type pseudo-differential operator of order 1, it follows that $e^{\sqrt{-1}t\mathbf{D}R}$ is a unitary Fourier integral operator whose canonical relation is given by

$$C = \{((w, \xi), (v, \zeta)) : (w, \xi), (v, \zeta) \in T^*X \setminus \{0\}, (w, \xi) = \exp tX_{\sigma_{\mathbf{D}R}}^{T^*X}(v, \zeta)\}$$

(see, e.g., [DG], Theorem 1.1, or [H], Theorem 29.1.1; note that ellipticity is not essential). It follows then from Lemma 6.4 that the canonical relation of $V(t)$ is given by the time t flow-out of Σ under the flow of the Hamiltonian $-\sigma_{\mathbf{D}R} = r\pi^*\dot{\varphi}_0$ with respect to (T^*X, ω_{T^*X}) . As shown in the proof of Proposition 5.2 this coincides with the time t flow of Σ under the flow of the same Hamiltonian with respect to (Σ, ω_Σ) . Finally, the corresponding statement for the operators $V_N(t)$ asserted in the Proposition follows by ‘de-homogenization’, since when restricting to $H^0(M, L^N)$, $N \in \mathbb{N}$, the operator \mathbf{D} simply acts by multiplication by N , and so we may replace r by the constant N , concluding the proof. \square

REMARK 6.5. In place of $\gamma_\epsilon(D_{\hat{f}}, \mathbf{D})$ we could use at least two other constructions.

First, we could replace $\dot{u}_0(D_{\hat{f}}\mathbf{D}^{-1})$ with the globally well-defined operator

$$\dot{u}_0(D_{\hat{f}}(I + |\mathbf{D}|^2)^{-1/2}).$$

We have that $s_{D_{\hat{f}}}(1 + |s_{\mathbf{D}}|^2)^{-1/2} \approx s_{D_{\hat{f}}}s_{\mathbf{D}}^{-1}$ asymptotically as $r \rightarrow \infty$ in Σ . Therefore, the principal symbols (that are homogeneous of degree 0) of the associated Toeplitz operators are equal, $\sigma_{\dot{u}_0(D_{\hat{f}}(I + |\mathbf{D}|^2)^{-1/2})}|_\Sigma = \sigma_{\dot{u}_0(D_{\hat{f}}\mathbf{D}^{-1})}|_\Sigma$. However, this new operator has different eigenvalues (although this would not matter later in proving Theorem 1, since $\frac{\alpha}{N} - \frac{\alpha}{\sqrt{1+N^2}} = O(1/N^3)$).

Alternatively, we could use the orthogonal projection Π_0^\perp onto the orthogonal complement of the invariant functions. This is finer than $1 - \chi(\mathbf{D})$ since its symbol vanishes outside $\{\sigma_{\mathbf{D}} = 0\}$ and not just outside an open cone containing it. Note that $\{\sigma_{\mathbf{D}} = 0\} \subset T^*X$ is the ‘dual’ of the horizontal bundle over X in TX (with respect to the connection α), and does not intersect Σ which is itself dual to the vertical bundle. Since the spectrum of \mathbf{D} lies in \mathbb{Z} , the operator $\eta\dot{u}_0(D_{\hat{f}}\mathbf{D}^{-1})\Pi_0^\perp$ is well-defined on all of $L^2(X)$. From the formula $\Pi_0 = \frac{1}{2\pi} \int_0^{2\pi} e^{\sqrt{-1}\theta\mathbf{D}} d\theta$, we see that Π_0 is a zeroth order Fourier integral operator whose canonical relation is

$$C = \{((w, \xi), (w', \xi')) \in T^*X \setminus \{0\} \times T^*X \setminus \{0\} :$$

$$\sigma_{\mathbf{D}}(w, \xi) = 0, w' = e^{\sqrt{-1}\theta}w, \xi' = e^{\sqrt{-1}\theta}\xi, \text{ for some } \theta \in [0, 2\pi)\}.$$

Since $\Pi_0^\perp = I - \Pi_0$ is also a Fourier integral operator, $(\eta\dot{u}_0)(D_{\theta}\mathbf{D}^{-1})\Pi_0^\perp$ is a well-defined Fourier integral operator and $\Pi\eta\dot{u}_0(D_{\theta}\mathbf{D}^{-1})\Pi_0^\perp\Pi = \Pi\eta\dot{u}_0(D_{\theta}\mathbf{D}^{-1})\Pi$, as Toeplitz Fourier integral operators. We can then compute the symbol of the Fourier

integral operator $\eta \dot{u}_0(D_\theta \mathbf{D}^{-1}) \Pi_0^\perp$ using the composition theorem [H] and obtain the same answer as in the proof of Lemma 6.4, since C has an empty composition with the canonical relation of Π , $\Delta_{\Sigma \times \Sigma} \subset \Sigma \times \Sigma$, as $\sigma_{\mathbf{D}}|_\Sigma \neq 0$.

6.3. Comparison of the quantizations. The reason we introduced the second quantization is that its eigenvalues are explicitly given in terms of the symplectic potential. Since both of our unitary one-parameter subgroups quantize the same Hamiltonian flow, we obtain the following relation between their spectra.

LEMMA 6.6. *We have*

$$\mu_{N,\alpha} = -\dot{u}_0\left(\frac{\alpha}{N}\right) + O(1/N).$$

More precisely, there exists $C > 0$ independent of N or $\alpha \in NP$ such that

$$|\mu_{N,\alpha} + \dot{u}_0(\alpha/N)| \leq \frac{C}{N}.$$

Proof. By Lemma 6.4, $\Pi_N \dot{\varphi}_0 \Pi_N$ and $-\Pi_N \dot{u}_0(D_{\hat{f}} \mathbf{D}^{-1}) \Pi_N$ are zeroth order Toeplitz operators with the same principal symbols. Hence they differ by a Toeplitz operator of order -1 . Let $\chi_\alpha \in H^0(M, L^N)$. It follows that

$$\mu_{N,\alpha} = \frac{\langle \Pi_N \dot{\varphi}_0 \Pi_N \chi_\alpha, \chi_\alpha \rangle}{Q_{h_0^N}(\alpha)} = -\frac{\langle \Pi_N \dot{u}_0(D_{\hat{f}} \mathbf{D}^{-1}) \Pi_N \chi_\alpha, \chi_\alpha \rangle}{Q_{h_0^N}(\alpha)} + O\left(\frac{1}{N}\right),$$

proving the Lemma. \square

REMARK 6.7. We briefly relate the Lemma above to some calculations in [SoZ1, Z4]. First, $\Pi_N \dot{u}_0(D_{\hat{f}} \mathbf{D}^{-1}) \Pi_N(z, z)$ is precisely the kind of Bernstein polynomial discussed in [Z4]. There it was shown that

$$\Pi_N \dot{u}_0(D_{\hat{f}} \mathbf{D}^{-1}) \Pi_N(z, z) = N^n \dot{u}_0(\nabla \psi_0(z)) + O(N^{n-1}) = -\dot{\varphi}_0(z) N^n + O(N^{n-1}).$$

In the language of Berezin-Toeplitz operators, this shows that $\Pi_N \dot{u}_0(D_{\hat{f}} \mathbf{D}^{-1}) \Pi_N$ and $-\Pi \dot{\varphi}_0 \Pi_N$ have the same Berezin symbol. There is an invertible (Berezin) transform from the Toeplitz symbol (calculated in Lemma 6.4) and the Berezin symbol, so this gives another proof of Lemma 6.6 (noting the N^n factor in passing from $H_N^2(X)$ to $H^0(M, L^N)$, see §§4.1).

One could also evaluate the eigenvalues directly by pushing forward the eigenvalue integral to P via the moment map $\nabla \psi_0$ and using equations (28),(37), and the identity

$$(\nabla \psi_0)_* \omega_{h_0}^n = dy,$$

giving

$$\mu_{N,\alpha} = \frac{1}{Q_{h_0^N}(\alpha)} \int_P -\dot{u}_0(y) e^{N(u_0(y) + \langle \frac{\alpha}{N} - y, \nabla u_0(y) \rangle)} dy. \quad (56)$$

Integrals similar to this one are calculated asymptotically in [SoZ1]. For instance, when $d(\frac{\alpha}{N}, \partial P) > \frac{\log N}{N}$ we may apply the steepest descent method to (56). There is a unique critical point $y = \frac{\alpha}{N}$ and

$$\mu_{N,\alpha} = -\frac{1}{Q_{h^N}(\alpha)} \dot{u}_0(\alpha/N) Q_{h^N}(\alpha) + O\left(\frac{1}{N}\right). \quad (57)$$

The evaluation in the boundary zone is more complicated and can be done by Taylor expansions centered at the boundary.

As a corollary of Lemma 6.6 we have a corresponding result on the level of potentials. Let $h_s = e^{-\varphi_s} h_0$.

COROLLARY 6.8. *There exists a constant $C > 0$ independent of N or z such that*

$$|\tilde{\varphi}_N(s, z) - \varphi_N(s, z)| \leq \frac{Cs \log N}{N}.$$

Proof. By Lemma 6.6, we have for some uniformly bounded function $R(N, \alpha)$ that

$$\begin{aligned} \varphi_N(s, z) &= \frac{1}{N} \log \sum_{\alpha} e^{sN\mu_{N,\alpha}} \frac{|\chi_{\alpha}|_{h_0^N}^2}{\mathcal{Q}_{h_0^N}(\alpha)} \\ &= \frac{1}{N} \log \sum_{\alpha} e^{-sN\dot{u}_0(\alpha/N) + sR(N,\alpha)} \frac{|\chi_{\alpha}|_{h_0^N}^2}{\mathcal{Q}_{h_0^N}(\alpha)}. \end{aligned} \quad (58)$$

The result now follows by comparing this with the expression (50) for $\tilde{\varphi}_N(s, z)$. \square

Equation (58) leads to a heuristic proof of Theorem 1: According to [SoZ1] (Propositions 3.1 and 6.1),

$$\mathcal{Q}_{h_0^N}(\alpha) = F(\alpha, N) e^{N\dot{u}_0(\alpha/N)} / N^{C(\alpha, n)}, \quad (59)$$

where $C(\alpha, n)$ and $F(\alpha, N)$ are some uniformly bounded functions. Substituting this into (58) we obtain

$$\varphi_N(s, z) = \frac{1}{N} \log \sum_{\alpha} e^{N(\langle x, \alpha/N \rangle - \psi_0(x) - u_s(\alpha/N)) + sR(N, \alpha)} + O(\log N/N). \quad (60)$$

Intuitively, the leading order logarithmic asymptotics are given by the value of the principal part of the exponent,

$$\langle x, \alpha/N \rangle - \psi_0(x) - u_s(\alpha/N),$$

at its maximum (over $\alpha \in NP \cap \mathbb{Z}^n$). But this value is $u_s^*(x) - \psi_0(x)$, as stated in Theorem 1.

In the next section we give a rigorous proof.

7. CONVERGENCE OF THE QUANTIZATION TO THE IVP GEODESIC

We now complete the proof of Theorem 1. We study the large N limit of

$$\varphi_N(s, z) = \frac{1}{N} \log \sum_{\alpha \in NP \cap \mathbb{Z}^n} e^{sN\mu_{N,\alpha}} \mathcal{P}_{h_0^N}(\alpha, z), \quad (61)$$

First note that the $C^2(M \times [0, T_{\text{span}}^{\text{cvx}}))$ convergence is a direct corollary of the work of Song-Zelditch [SoZ1]. Indeed for all $T < T_{\text{span}}^{\text{cvx}}$ the geodesic is smooth and so we may consider it as a smooth endpoint geodesic connecting φ_0 to $\varphi_T \in \mathcal{H}(\mathbf{T})$. It thus

remains to prove that $\varphi_N(s, z)$ converges to $\varphi_s(z)$ in $C^0(M \times [0, T])$ for all $T > 0$. The argument here is somewhat different than the corresponding C^0 convergence results in [RZ1, SoZ1, SoZ2] due to the fact that our limit is less regular, namely only Lipschitz. Due to this reduced regularity we may not apply asymptotic expansions for families of smooth Bergman metrics (for example the asymptotic expressions for the norming constants or the peak values of the monomials derived in [SoZ1]), nor can we use the standard asymptotic expansion of the Bergman kernel [Z3]. Finally, we also do not have a genuine moment map.

According to Corollary 6.8, in order to prove convergence of $\varphi_N(s, z)$ to $\varphi_s(z)$ it will be enough to consider the difference

$$E_N(s, z) := \tilde{\varphi}_N(s, z) - \varphi_s(z) = \frac{1}{N} \log \sum_{\alpha \in NP \cap \mathbb{Z}^n} e^{-sN\dot{u}_0(\alpha/N)} \frac{|\chi_\alpha(z)|_{h_s^N}^2}{\mathcal{Q}_{h_0^N}(\alpha)} \quad (62)$$

Theorem 1 will then follow from the following result.

LEMMA 7.1. *For every $T > 0$, we have*

$$\lim_{N \rightarrow \infty} \sup_{s \in [0, T]} \|E_N(s, z)\|_{C^0(M)} = 0. \quad (63)$$

Proof. Whenever $T < T_{\text{span}}^{\text{cvx}}$ the result follows directly from (59) and the asymptotic expansion of the Bergman kernel: applying (59) to h_0 , using the explicit formula for u_s and then applying (59) to h_s , we obtain

$$E_N(s, z) = \frac{1}{N} \log \sum_{\alpha \in NP \cap \mathbb{Z}^n} \frac{|\chi_\alpha(z)|_{h_s^N}^2}{\mathcal{Q}_{h_s^N}(\alpha)} + O(\log N/N),$$

and this is $O(\log N/N)$ by the asymptotic expansion of the Bergman kernel [Z3]. Here by $O(\log N/N)$ we mean a quantity that is bounded from above and below by $\pm C \frac{\log N}{N}$ where C may depend on the Cauchy data and on T .

Assume now that $T \geq T_{\text{span}}^{\text{cvx}}$. First, we have

$$e^{-sN\dot{u}_0(\alpha/N)} \frac{|\chi_\alpha(z)|_{h_s^N}^2}{\mathcal{Q}_{h_0^N}(\alpha)} = e^{-sN\dot{u}_0(\alpha/N)} \frac{e^{\langle x, \alpha \rangle - N\psi_s}}{\mathcal{Q}_{h_0^N}(\alpha)}. \quad (64)$$

From the definition of the Legendre transform we obtain that this is bounded from above by

$$e^{-sN\dot{u}_0(\alpha/N) + \langle x, \alpha \rangle + Nu_s(\alpha/N) - N\langle x, \alpha/N \rangle} / \mathcal{Q}_{h_0^N}(\alpha) = e^{Nu_0(\alpha/N)} / \mathcal{Q}_{h_0^N}(\alpha).$$

Applying (59) to h_0 and using the fact that $\dim H^0(M, L^N)$ is polynomial in N we obtain that

$$E_N(s, z) \leq O(\log N/N).$$

We now turn to proving a lower bound for $E_N(s, z)$ when $T \geq T_{\text{span}}^{\text{cvx}}$. Rewrite (64) as

$$e^{\langle x, \alpha \rangle - N\psi_s - Nu_s(\alpha/N)} F(\alpha, N) N^{-C}.$$

A lower bound for $E_N(s, z)$ will follow once we find one summand in (62) that is not decaying to zero too fast. More precisely, we will seek $\tilde{N} = \tilde{N}(s, x)$ and one $\alpha = \alpha(N, s, x) \in NP \cap \mathbb{Z}^n$ for each $N > \tilde{N}$, for which

$$e^{\langle x, \alpha \rangle - N\psi_s - Nu_s(\alpha/N)} \geq e^{-CN^{1-\epsilon}},$$

for some $\epsilon > 0$.

Fix $x \in \mathbb{R}^n$ (recall $|z|^2 = e^x$). The Kähler potential ψ_s is defined on all of \mathbb{R}^n and satisfies

$$\psi_s(x) \geq \langle x, y \rangle - u_s(y), \quad \forall y \in P, \quad (65)$$

with equality if and only if $y \in \partial\psi_s(x)$ (see [Ro]). Let $y_1 \in P$ satisfy equality in (65). It exists, since the supremum in

$$\psi_s(x) = \sup_{y \in P} [\langle x, y \rangle - u_s(y)],$$

is necessarily achieved and finite (P is compact and u_s is bounded); hence by convexity of ψ_s we have $\partial\psi_s(x) \neq \emptyset$, and one may choose then $y_1 \in \partial\psi_s(x)$. Then we need to find $\tilde{N} = \tilde{N}(s, x)$ and $\alpha = \alpha(N, s, x)$ such that

$$e^{N(\langle x, \alpha/N - y_1 \rangle + u_s(y_1) - u_s(\alpha/N))} \geq e^{-CN^{1-\epsilon}}, \quad \text{for each } N > \tilde{N}.$$

In fact we will derive such an estimate where the right hand side is $e^{-C \log N}$. First, we need the following result concerning $\partial\psi_s(x)$.

CLAIM 7.2. *Let $x \in \mathbb{R}^n$ and let $y_1 \in \partial\psi_s(x)$. Then $y_1 \in P \setminus \partial P$.*

Proof. Note that by duality $x \in \partial u_s(y_1)$ (this holds even though u_s need not be convex, see [HL], Theorem 1.4.1, p. 47), and in particular $\partial u_s(y_1) \neq \emptyset$. Therefore, it suffices to show that $\lim_{y \rightarrow \partial P} |\nabla u_s(y)| = \infty$, since that will imply that $\partial u_s(y) = \emptyset$ whenever $y \in \partial P$.

Let $\{w_i\} \subset P \setminus \partial P$ be a sequence converging to $y \in \partial P$. Assume without loss of generality that l_1, \dots, l_n provide a coordinate chart in a neighborhood of y in P . Using Guillemin's formula (39), in these coordinates the gradient of u_s takes the form $(\log l_1 + h_1, \dots, \log l_n + h_n)$, where $h_j \in C^\infty(P)$, $j = 1, \dots, n$. It then follows that $\lim_{y \rightarrow \partial P} |\nabla u_s(y)| = \infty$, as desired. \square

The points $\{\alpha/N\}_{NP \cap \mathbb{Z}^n}$ are C/N -dense in P , where $C > 0$ is some uniform constant. Hence, for each of the 2^n orthants in \mathbb{R}^n there exists a point α/N that is C/N -close to y_1 and such that the vector $\alpha/N - y_1$ is contained in that orthant. Now let \tilde{N} be chosen large enough so that $\text{dist}(y_1, \partial P) > C/\tilde{N}$ (possible by Claim 7.2). Further, let \tilde{N} be chosen so large such that we may find $\alpha_1 = \alpha_1(\tilde{N})$ such that $\alpha_1/\tilde{N} \in P \setminus \partial P$ and

$$\text{dist}(\alpha_1/\tilde{N}, \partial P) > C/\tilde{N}, \quad (66)$$

and also

$$\langle \alpha_1/\tilde{N} - y_1, x \rangle \geq 0, \quad \text{and} \quad \frac{C}{2\tilde{N}} \leq |\alpha_1/\tilde{N} - y_1| \leq \frac{C}{\tilde{N}}. \quad (67)$$

Note that y_1 depends only on s and x and so does \tilde{N} . Further, for every $N > \tilde{N}$ one may find an $\alpha_1 = \alpha_1(N)$ satisfying the inequalities (66) and (67) with \tilde{N} replaced by N .

Applying the mean value theorem to the line segment between α_1/N and y_1 , it follows that

$$e^{N(\langle x, \alpha_1/N - y_1 \rangle + u_s(y_1) - u_s(\alpha_1/N))} \geq e^{-N|y_1 - \alpha_1/N| |\nabla u_s(y_2)|}, \quad (68)$$

where $y_2 \in P \setminus \partial P$ is some point on the line segment between α_1/N and y_1 . Hence,

$$\text{dist}(y_2, \partial P) > C/N.$$

By Guillemin's formula (39), we therefore have

$$|\nabla u_s(y_2)| < C \log N + s \|\dot{u}_0\|_{C^1(P)} < C_T \log N,$$

for some constant C_T that depends on T . It follows that

$$\begin{aligned} E_N(s, z) &\geq \frac{1}{N} \log e^{N(\langle x, \alpha_1(N)/N - y_1 \rangle + u_s(y_1) - u_s(\alpha_1(N)/N))} \\ &\geq \frac{1}{N} \log e^{-C_T \log N} \geq \frac{-C_T \log N}{N}, \end{aligned} \quad (69)$$

and this concludes the proof of Lemma 7.1. \square

Lemma 7.1 completes the proof of Theorem 1.

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